**Report outline:**

1. Introduction:
   1. What’s a wave motion and what’s a water wave problem; the relevance and the difficulty;
   2. Introduce the well-known results on the whole line (wave, KdV equations)
   3. Mention the contribution of capstone – derivation of approximate equations on the whole line
2. Preliminaries + Literature review:
   1. Explain the physical assumptions made in the water-wave problem (and justify them in the context of water waves), Lannes 2013
   2. Explain the shallow water regime and why we care about it:
   3. Explain what’s a wave equation, KdV, discuss their history, importance
   4. Explain time scales & explain up to what time the expansion is valid
   5. Explain why we can trust asymptotic models (i.e. justify that solutions of asymptotic model are asymptotic to actual solutions in the shallow water)b7
3. Derivation from non-local formulation on the whole line
   1. Explain why we care about the non-local formulation (what are the pros, cons, etc) in general:
      1. Simplify equations of motion
   2. Explain why we care about our particular formulation, Oliveras & Vasan 2013
      1. In physical variables, derived without approximation
      2. works for both 1 and 2 dimensional surfaces
      3. in eta only, so that’s good for applications ,
   3. Derive the non-local formulation (use Report 2)
   4. Derive the surface expression (use Report 2)
   5. Derive wave and KdV equations (use Report 2)
      1. Make sure to explain that equations are non-dimensional, and so as we dispense with epsilon, the relationship between equations is hidden
   6. Provide justification for the model: why do we care about obtaining wave and KdV from this formulation?
   7. Discuss alternative derivations and pros and cons (if any). Alternative derivations:
      1. (Ablowitz et al 2006) – the derivation is in velocity potential only
      2. (BBM 1972) – the KdV is derived by combining approximate dispersion and nonlinear equation, but there is no reason to assume that dispersion and nonlinearity have such a balance
      3. DNO (Craig & Sulem 1993) – approximate equations are not derived
4. Future directions:
   1. Discussion of the half-line problem and associated difficulty (emphasise the open-ended nature of the problem)
   2. Moving to bounded domains

**References:**

Oliveras, Vasan 2013, A new equation describing travelling water waves

Ablowitz, Fokas, Musslimani 2016, On a new nonlocal formulation of water waves

Craig, Sulem 1993, Numerical simulation of gravity waves

David Lannes 2013 The water waves problem

Ablowitz 2011, Nonlinear dispersive waves

We note that the equations of motion are challenging to work with directly, due to the nonlinear boundary conditions, as well as the unknown domain. In addition to asymptotics, these complications cause difficulties when attempting to deal with questions of well-posedness and existence. To address these issues, reformulations of the problem have been introduced: they result in equivalent problems that are more tractable to investigating asymptotics, existence, etc. While such reformulations can be helpful, they may suffer from other issues. Below, we give a short overview of these formulations, along with explaining the drawbacks. Our main goal is to look for a reformulation that’s focused on the water surface, and generalises easily to two-dimensional problem. This is because in applications, determining the water surface is the main interest.

For example, for one-dimensional surfaces (no $y$ variable), conformal mappings have been used to eliminate these problems (for an overview, see \cite{DKSZ}). However, the conformal mapping approach does not generalize to two-dimensional surfaces.

For both one- and two-dimensional surfaces, other formulations (such as the Hamiltonian formulation given in Zakharov 1968 or the Zakharov–Craig–Sulem formulation, Craig & Sulem 1993) reduce the Euler equations to a system of two equations, in terms of surface variables $q$ and $\eta$ only, by introducing a Dirichlet-to-Neumann operator (DNO). However, using their formulation one must truncate the series expansion of the DNO. The non-local formulation presented below does not require such a truncation.

In a similar spirit, Ablowitz, Fokas & Musslimani (2006) introduce a new non-local formulation of Euler’s equation (henceforth referred to as the AFM formulation) that results in a system of two equations for the same variables as presented in the DNO formulation. Both the DNO and AFM formulations reduce the problem from the full fluid

domain to a system of equations that depend on the surface elevation  and the

velocity potential evaluated at the surface q.x/, where

q.x/ D .x; .x//: (1.9)

While this simplification significantly reduces the computational domain, one could

argue that the equations require solving for an additional function q, which is typically

of less interest and not easily measured in experiments. The primary interest in

applications is determining the surface elevation.

What’s more important: eta or q or phi?